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NOTE

ON THE STRUCTURE OF TRIANGLE-FREE GRAPHS OF LARGE MINIMUM DEGREE

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It is shown that for every $\varepsilon > 0$ there exists a constant L such that every triangle-free graph on n vertices with minimum degree at least $(1/3+\varepsilon)n$ is homomorphic to a triangle-free graph on at most L vertices.

Let $\mathcal{G}_n(\varepsilon)$ denote the set of all triangle-free graphs on n vertices labelled by natural numbers of minimum degree larger than $(1/3+\varepsilon)n$; let also $\mathcal{G}_n = \mathcal{G}_n(0)$. The family $\mathcal{G}_n$ has been studied by a few authors (see Brandt [2] for a brief survey on results and open problems on $\mathcal{G}_n$). Recently, Thomassen [7] proved that for every $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that the chromatic number of every $G \in \mathcal{G}_n(\varepsilon)$ is smaller than K. On the other hand, for every $\varepsilon > 0$ and N, Hajnal constructed triangle-free graphs G of minimum degree larger than $(1/3-\varepsilon)n$ with the chromatic number $\chi(G) > N$ (see Erdős and Simonovits [4]). In this note we supplement Thomassen's theorem, setting in the affirmative a conjecture of Jin [5] (see also Question 1 in [7]). We say that a graph $H = (V,E)$ is *homomorphic* to $H' = (V',E')$ if there exists a function (homomorphism) $\psi : V \to V'$ such that $\{\psi(v), \psi(w)\} \in E'$ whenever $\{v,w\} \in E$.

Theorem 1. *For every $\varepsilon > 0$ there exists $L = L(\varepsilon)$ such that each $G \in \mathcal{G}_n(\varepsilon)$ is homomorphic to some triangle-free graph on at most L vertices.*

We remark that although the above result implies that $\chi(G) \le L(\varepsilon)$ for each $G \in \mathcal{G}_n(\varepsilon)$, our estimates for $L(\varepsilon)$ are by far worse than the bound $K(\varepsilon) = O(1/\varepsilon)$ given by Thomassen in [7].

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In the proof of Theorem we shall use the following two simple observations.

Claim 2. *If S_1, S_2 are two independent sets of $G \in \mathcal{G}_n(\varepsilon)$ such that $S_1 \cap S_2 \neq \emptyset$, then $|S_1 \cup S_2| < (2/3 - \varepsilon)n$.*

Proof. It is enough to note that the neighbourhood of a vertex $v \in S_1 \cap S_2$ must be disjoint with $S_1 \cup S_2$. ∎

Claim 3. *Each two non-adjacent vertices v_1, v_2 of a maximal triangle-free graph $G \in \mathcal{G}_n(\varepsilon)$ share more than $3\varepsilon n$ common neighbours.*

Proof. Since G is maximal triangle-free, v_1 and v_2 have a common neighbour v. But then the neighbourhood of each of v_1 and v_2 must be disjoint from the neighbourhood of v. ∎

We shall also need the following result of Brandt [2]. Here and below Q_3 denote the three-dimensional cube, i.e., a graph obtained from the complete bipartite graph $K_{4,4}$ by removing a perfect matching.

Lemma 4. *A maximal triangle-free graph $G \in \mathcal{G}_n$ does not contain a copy of Q_3 as an induced subgraph.* ∎

Proof of Theorem. Clearly, it is enough to show the assertion only for a maximal triangle-free graphs which belong to $\mathcal{G}_n(\varepsilon)$, where $0 < \varepsilon < 0.01$ and n is large enough. Let $G = (V, E)$ be such a graph. Using Szemerédi's Regularity Lemma [6] (see also monographs of Bollobás [1] and Diestel [3]), we infer that there exists a constant $M_0 = M_0(\varepsilon)$, which depends only on ε but not on G, so that one can partition the vertex set V of G into sets $V_0, V_1, \ldots, V_M$ such that

(i) $\varepsilon^{-10} \leq M \leq M_0$;
(ii) $|V_0| \leq \varepsilon^{10} n$ and $|V_1| = |V_2| = \cdots = |V_M|$;
(iii) all but at most $\varepsilon^{10}\binom{M}{2}$ pairs $\{V_i, V_j\}$, where $1 \leq i < j \leq M$, are ε^{10}-regular.

Our first goal is to modify this partition, moving some vertices from $\bigcup_{i=1}^{M} V_i$ to V_0, to obtain a new partition $\bar{V}_0, \bar{V}_1, \ldots, \bar{V}_{\bar{M}}$ for some $\bar{M} \leq M$ such that

1. $\bar{V}_0 \supseteq V_0$ and $|\bar{V}_0| \leq 2\varepsilon^3 n$;
2. $|\bar{V}_1| = |\bar{V}_2| = \cdots = |\bar{V}_{\bar{M}}| = \lceil (1 - \varepsilon^3)|V_1| \rceil$, and for each $i = 1, \ldots, \bar{M}$, there exists $j = 1, \ldots, M$, such that $\bar{V}_i \subseteq V_j$;
3. the sets $\bar{V}_i$, $i = 1, \ldots, \bar{M}$, are independent;

4. for every $1 \le i < j \le M$, the pair $\{\bar{V}_i, \bar{V}_j\}$ is joined by either no edges, or by $|\bar{V}_i||\bar{V}_j|$ of them (i.e., its density is either 0 or 1);
5. the graph $\bar{G}$ induced in G by $\bigcup_{i=1}^{\bar{M}} \bar{V}_i$ is homomorphic to a maximal triangle-free graph $\bar{H} \in \mathcal{G}_{\bar{M}}(0.9\varepsilon)$.

Note first that from (iii) it follows that for all but at most $2\varepsilon^5 M$ 'bad' sets V_i there exist fewer than $2\varepsilon^5 M$ sets V_j such that the pair $\{V_i, V_j\}$ is not ε^{10}-regular. We move all the vertices of the bad sets to V_0, and call the resulting partition $\hat{V}_0, \hat{V}_1, \dots, \hat{V}_{\hat{M}}$. Observe also that, if n is large enough, then every ε^{10}-regular pair $\{\hat{V}_i, \hat{V}_j\}$ with density ρ, where $\varepsilon^3 \le \rho \le 1-\varepsilon^9$, contains an induced copy of Q_3, while, by Lemma 4, no such copy can appear in G.

Now let us make the following observation.

Claim 5. *Let us assume that for some given i, j, $1 \le i < j \le \hat{M}$, at least $5\varepsilon^2|\hat{V}_i||\hat{V}_j|$ pairs of vertices $\{w_i, w_j\}$, $w_i \in \hat{V}_i$, $w_j \in \hat{V}_j$, are not edges of G. Then there exists k, $1 \le k \le \hat{M}$, such that each of the pairs $\{\hat{V}_i, \hat{V}_k\}$ and $\{\hat{V}_j, \hat{V}_k\}$ is ε^{10}-regular and has density at least $1-\varepsilon^9$.*

Proof. Let us choose greedily $r_0 = \lceil 2\varepsilon^2|\hat{V}_i|\rceil$ disjoint pairs $\{w_i^r, w_j^r\}$, such that $w_i^r \in \hat{V}_i$, $w_j^r \in \hat{V}_j$, and the pair $\{w_i^r, w_j^r\}$ is not an edge of G for each $r = 1, 2, \dots, r_0$. Claim 3 implies that for each $r = 1, 2, \dots, r_0$, the vertices w_i^r, w_j^r have a lot of common neighbours, among which at least $3\varepsilon n - |\hat{V}_0| - \varepsilon^4 n \ge 2\varepsilon n$ belong to sets $\hat{V}_k$ such that both pairs $\{\hat{V}_i, \hat{V}_k\}$ and $\{\hat{V}_j, \hat{V}_k\}$ are ε^{10}-regular. Hence, by elementary counting argument, there is a $\ell \ne i, j$, $1 \le \ell \le \hat{M}$, such that both pairs $\{\hat{V}_i, \hat{V}_\ell\}$ and $\{\hat{V}_j, \hat{V}_\ell\}$ are ε^{10}-regular and have density larger than ε^3. Since, as we have observed above, each ε^{10}-regular pair of density larger than ε^3 has, in fact, density at least $1-\varepsilon^9$, the assertion follows. ∎

From Claim 5 it follows that a pair $\{\hat{V}_i, \hat{V}_j\}$ can be only either very dense or very sparse. Thus, we call a pair $\{\hat{V}_i, \hat{V}_j\}$ dense if it is joined by more $(1-5\varepsilon^2)|\hat{V}_i||\hat{V}_j|$ edges; note that for no triple $\{\hat{V}_i, \hat{V}_j, \hat{V}_k\}$ all three pairs $\{\hat{V}_i, \hat{V}_j\}$, $\{\hat{V}_i, \hat{V}_k\}$, $\{\hat{V}_j, \hat{V}_k\}$ are dense, since such a 'dense triple' would lead to a triangle in G. If a pair $\{\hat{V}_i, \hat{V}_j\}$ is not dense, then, by Claim 5, there exists V_k such that both pairs $\{\hat{V}_i, \hat{V}_k\}$ and $\{\hat{V}_j, \hat{V}_k\}$ are dense and ε^{10}-regular, and thus, to avoid a triangle, the sets $\hat{V}_i$, $\hat{V}_j$ are joined by fewer than $\varepsilon^9|\hat{V}_i||\hat{V}_j|$ edges; in such a case we say that the pair $\{\hat{V}_i, \hat{V}_j\}$ is sparse.

Now we shall deal with 'missing links', i.e., pairs $\{w_i, w_j\}$ which are not edges of G but $w_i \in \hat{V}_i$, $w_j \in \hat{V}_j$, where the pair $\{\hat{V}_i, \hat{V}_j\}$ is dense. We move such vertices to $\hat{V}_0$ one by one. Note that, by Claim 3, w_i, w_j share at least $3\varepsilon n$ common neighbours, from which at least $2\varepsilon n$ do not belong to $\hat{V}_0 \cup \hat{V}_i \cup \hat{V}_j$.

Furthermore, since there are no dense triples $\{\hat{V}_i, \hat{V}_j, \hat{V}_k\}$, at least εn of the edges adjacent to either w_i or w_j belong to sparse pairs (i.e., they join two sets $\hat{V}_r$, $\hat{V}_s$, such that the pair $(\hat{V}_r, \hat{V}_s)$ is sparse). Thus, moving both w_i and w_j to $\hat{V}_0$, we decrease the number of edges which belong to sparse pairs by at least εn. Since the total number of such edges is smaller than $\varepsilon^9 n^2$, we shall remove all missing links from the graph in at most $\varepsilon^8 n$ steps. Finally, we 'balance' the resulting partition $\tilde{V}_0, \tilde{V}_1, \dots, \tilde{V}_{\hat{M}}$ by removing all the vertices which belong to sets $\tilde{V}_i$ with fewer than $(1-\varepsilon^3)|V_1|$ vertices to $\tilde{V}_0$, and decrease the size of all the remaining sets $\tilde{V}_i$ to $\lceil (1-\varepsilon^3)|V_1| \rceil$ by moving some of their vertices to $\tilde{V}_0$.

In the partition $\bar{V}_0, \bar{V}_1, \dots, \bar{V}_{\bar{M}}$ obtained in this way all dense pairs become, in fact, complete bipartite graphs. Since each set $\bar{V}_i$, $i = 1, 2, \dots, \bar{M}$, belongs to more than $\bar{M}/3$ dense pairs, and G is triangle-free, each such $\bar{V}_i$ is independent. Suppose that for some i, j, $1 \le i \le j \le \bar{M}$, the bipartite subgraph induced by $\{\bar{V}_i, \bar{V}_j\}$ is not complete, i.e., some vertices $w_i \in \bar{V}_i$, $w_j \in \bar{V}_j$, are not adjacent. Since the subgraph $\bar{G}$ induced in G by $\bigcup_{i=1}^{\bar{M}} \bar{V}_i$ contains all but at most $2\varepsilon^3 n$ vertices of G, from Claim 3 it follows that w_i, w_j have a common neighbour $w_k \in \bar{V}_k$ such that both pairs $\{\bar{V}_i, \bar{V}_k\}$, $\{\bar{V}_j, \bar{V}_k\}$, are dense and thus complete. Consequently, since G is triangle-free, the subgraph induced in G by $\bar{V}_i \cup \bar{V}_j$ contains no edges. A similar argument shows that a graph $\bar{H}$ with vertex set $\{\bar{V}_1, \dots, \bar{V}_{\bar{M}}\}$, whose set of edges consists of all dense pairs $\{\bar{V}_i, \bar{V}_j\}$, is a maximal triangle-free graph from $\mathcal{G}_{\bar{M}}(0.9\varepsilon)$. Hence, (1)–(5) hold.

In order to complete the proof we need to study the structure of the subgraph induced in G by the set of the 'abandoned' vertices $\bar{V}_0$. For each vertex w of $\bar{V}_0$ denote by $S(w)$ the set of at least $(1/3 + 0.9\varepsilon)n$ neighbours of w in $\bigcup_{i=1}^{\bar{M}} \bar{V}_i$, and let $S_H(w)$ be the set of all $\bar{V}_i$'s, $i = 1, 2, \dots, \bar{M}$, such that at least one vertex from $\bar{V}_i$ is adjacent to w. Note that $S_H(w)$ is an independent subset of $\bar{H}$, for every $w \in \bar{V}_0$.

Observe now that if $S_H(w) \cap S_H(w') \neq \emptyset$, then Claim 2 and the fact that $\bar{H} \in \mathcal{G}_{\bar{M}}(0.9\varepsilon)$ imply that $|S_H(w) \cup S_H(w')| < (2/3 - 0.9\varepsilon)\bar{M}$. But then also $|S(w) \cup S(w')| < (2/3 - 0.8\varepsilon)n$, and, since $|S(w)|, |S(w')| \ge (1/3 + 0.9\varepsilon)n$, the vertices w and w' must have a common neighbour and thus be non-adjacent. On the other hand, if $S_H(w) \cap S_H(w') = \emptyset$, then also $S(w) \cap S(w') = \emptyset$, and since the neighbourhoods of w and w' have at most $|\bar{V}_0| < 3\varepsilon n$ elements in common, by Claim 3, w and w' are adjacent. Hence, if we partition the vertices w of $\bar{V}_0$ according to the 'index sets' $S_H(w)$, then all sets of the partition are independent and the structure of the subgraph they induced in G is uniquely determined by the structure of $\bar{H}$. Consequently, since the number of independent sets in $\bar{H}$ can be crudely bounded above by $2^{\bar{M}} \le 2^{M_0(\varepsilon)}$, the assertion follows with $L(\varepsilon) = M_0(\varepsilon) + 2^{M_0(\varepsilon)}$. ∎

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